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Fuzzy Vector Spaces and Fuzzy Topological Vector Spaces

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1. INTRODUCTION

The concept of a fuzzy set was introduced by Zadeh [11]. Several other authors have applied fuzzy sets to various branches of mathematics. In this paper we apply the concept of a fuzzy set to the elementary theory of vector spaces and topological vector spaces. Our notation and terminology for fuzzy sets follows that of [11]. For the definitions of fuzzy topological spaces, product and quotient spaces we follow Wong [9, 10] and Chang [2].

2. SUMS AND SCALAR PRODUCTS OF FUZZY SETS

Throughout this paper E will denote a vector space over K, where K is the space of real or complex numbers.

DEFINITION. Let $A_1, \dots, A_n$ be fuzzy sets in E. We define $A_1 \times \cdots \times A_n$ to be the fuzzy set A in E^n whose membership function is given by

$$\mu_A(x_1, \dots, x_n) = \min\{\mu_{A_1}(x_1), \dots, \mu_{A_n}(x_n)\}.$$

Let $f: E^n \to E, f(x_1, \dots, x_n) = x_1 + \cdots + x_n$. We define $A_1 + \cdots + A_n = f(A)$.
For λ a scalar and B a fuzzy set in E, we define $\lambda B = g(B)$ where $g: E \to E$, $g(x) = \lambda x$.

LEMMA 2.1. (a) *For $\lambda \neq 0$, $\mu_{\lambda B}(x) = \mu_B((1/\lambda)\, x)$ for all $x \in E$. For $\lambda = 0$,*

$$\mu_{\lambda B}(x) = 0, \qquad x \neq 0,$$
$$= \sup_y \mu_B(y), \qquad x = 0,$$

(b) *For all scalars λ and all $x \in E$ we have $\mu_{\lambda B}(x) \geq \mu_B(x)$.*

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PROPOSITION 2.2. *Let E, F be vector spaces over K, and let f be a linear map from E into F. Then, for all fuzzy sets A, B in E and all scalars λ,*

(a) $f(A+B) = f(A) + f(B)$;

(b) $f(\lambda A) = \lambda f(A)$.

Proof. (a) Let $M = \{f(x): x \in E\}$ and $w \in F$. We want to show that $a = b$, where $a = \mu_{f(A+B)}(w)$, $b = \mu_{f(A)+f(B)}(w)$. Assume first that $w \notin M$. Then $a = 0$. Also, if $x, y \in F$ with $x + y = w$, then at least one of the x, y is not in M and thus $\min\{\mu_{f(A)}(x), \mu_{f(B)}(y)\} = 0$. It follows that, for $w \notin M$, we have $b = 0 = a$. Assume next that $w \in M$. Given $\epsilon > 0$, there exists $z \in E$, with $f(z) = w$, such that $\mu_{A+B}(z) > a - \epsilon$. Hence there are $x, y \in E$, with $x + y = z$, such that

$$\min\{\mu_A(x), \mu_B(y)\} > a - \epsilon.$$

Since $f(x) + f(y) = w$, we have

$$\begin{aligned} b &\geqslant \min\{\mu_{f(A)}(f(x)), \mu_{f(B)}(f(y))\} \\ &\geqslant \min\{\mu_A(x), \mu_B(y)\} > a - \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we get $b \geqslant a$. On the other hand, given $\epsilon > 0$, there exist z_1, z_2 in F, with $z_1 + z_2 = w$, such that

$$b - \epsilon < \min\{\mu_{f(A)}(z_1), \mu_{f(B)}(z_2)\}.$$

Taking $\epsilon < b$ (if $b = 0$, then $a = 0$ and we have nothing to prove), we have that z_1, $z_2 \in M$. Therefore, there are x_1, x_2 in E, with $f(x_1) = z_1$ and $f(x_2) = z_2$, such that

$$b - \epsilon < \min\{\mu_A(x_1), \mu_B(x_2)\}.$$

Since $f(x_1 + x_2) = w$, we get $a > b - \epsilon$ and hence $a \geqslant b$ because $\epsilon > 0$ was arbitrary. This proves (a).

(b) Let $w \in F$, $c = \mu_{\lambda f(A)}(w)$, $d = \mu_{f(\lambda A)}(w)$. If $w \notin M$, then $c = d = 0$. Suppose that $w \in M$. If $\lambda \neq 0$,

$$\begin{aligned} c = \mu_{f(A)}(1/\lambda w) &= \sup_{f(x)=(1/\lambda)w} \mu_A(x) = \sup_{f(\lambda x)=w} \mu_{\lambda A}(\lambda x) \\ &= \sup_{f(y)=w} \mu_{\lambda A}(y) = d. \end{aligned}$$

Next assume that $\lambda = 0$. If $w \neq 0$, then $c = 0$. Also

$$d = \sup_{f(x)=w} \mu_{0A}(x) = 0$$

since, when $f(x) = w \neq 0$, $x \neq 0$. For $w = 0$, we have

$$c = \sup_{x \in F} \mu_{f(A)}(x) = \sup_{y \in E} \mu_A(y);$$

$$d = \sup_{f(x)=0} \mu_{0A}(x) = \mu_{0A}(0) = \sup_{y \in E} \mu_A(y).$$

This completes the proof.

COROLLARY 2.3. *$\lambda(A + B) = \lambda A + \lambda B$ for all fuzzy sets A, B in E and all scalars λ.*

We omit the proof of the following easily established lemma.

LEMMA 2.4. *Let $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ be fuzzy sets in E and put $A = A_1 + \cdots + A_n$, $B = B_1 + \cdots + B_m$, $F = A_1 + \cdots + A_n + B_1 + \cdots + B_m$. Then $F = A + B$.*

An ordinary subset A of E can be considered as a fuzzy set with membership function equal to the characteristic function of A. In this way we may consider sums of the form $A + B$ where one (or both) of the A, B is an ordinary subset of E. For $x \in E$ and B a fuzzy set, we define $x + B = \{x\} + B$.

Let $f_x \colon E \to E$, $f_x(y) = x + y$. One can easily show the following lemma.

LEMMA 2.5. *Let A be an ordinary subset of E and let B be a fuzzy set.*

Then

(i) $x + B = f_x(B)$;

(ii) $\mu_{x+B}(z) = \mu_B(z - x)$;

(iii) $A + B = \bigcup_{x \in A} (x + B)$.

PROPOSITION 2.6. *Let $A, A_1, \ldots, A_n$ be fuzzy sets in E and $\lambda_1, \ldots, \lambda_n$ scalars. The following assertions are equivalent.*

(1) $\lambda_1 A_1 + \cdots + \lambda_n A_n \subset A$.

(2) *For all $x_1, \ldots, x_n$ in E, we have*

$$\mu_A(\lambda_1 x_1 + \cdots + \lambda_n x_n) \geqslant \min\{\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n)\}.$$

Proof. (1) $\Rightarrow$ (2)

$$\begin{aligned} \mu_A(\lambda_1 x_1 + \cdots + \lambda_n x_n) &\geqslant \mu_{\lambda_1 A_1 + \cdots + \lambda_n A_n}(\lambda_1 x_1 + \cdots + \lambda_n x_n) \\ &\geqslant \min\{\mu_{\lambda_1 A_1}(\lambda_1 x_1), \ldots, \mu_{\lambda_n A_n}(\lambda_n x_n)\} \\ &\geqslant \min\{\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n)\}. \end{aligned}$$

(2) $\Rightarrow$ (1) By rearranging the order if necessary, we may assume that $\lambda_i \neq 0$ for $i = 1,\ldots, k$, and $\lambda_i = 0$ for $k < i \leqslant n$. Let $x_1 ,\ldots, x_k$ be elements of E. For all $y_1 ,\ldots, y_{n-k}$ in E we have

$$\mu_A(\lambda_1 x_1 + \cdots + \lambda_k x_k) \geqslant \min\{\mu_{A_1}(x_1),\ldots, \mu_{A_k}(x_k), \mu_{A_{k+1}}(y_1),\ldots, \mu_{A_n}(y_{n-k})\}.$$

Since $\mu_{0A_j}(0) = \sup_{y\in E} \mu_{A_j}(y)$, we get

$$\mu_A(\lambda_1 x_1 + \cdots + \lambda_k x_k) \geqslant \min\{\mu_{A_1}(x_1),\ldots, \mu_{A_k}(x_k), \mu_{0A_{\alpha+1}}(0),\ldots, \mu_{0A_n}(0)\}.$$

Now,

$$\begin{aligned}
&\mu_{\lambda_1 A_1+\cdots+\lambda_n A_n}(z)\\
&\quad = \sup_{x_1+\cdots+x_k=z} [\min\{\mu_{\lambda_1 A_1}(x_1),\ldots, \mu_{\lambda_n A_n}(x_n)\}]\\
&\quad = \sup_{x_1+\cdots+x_k=z} [\min\{\mu_{\lambda_1 A_1}(x_1),\ldots, \mu_{\lambda_k A_k}(x_k), \mu_{0A_{k+1}}(0),\ldots, \mu_{0A_n}(0)\}]\\
&\quad = \sup_{x_1+\cdots+x_k=z} [\min\{\mu_{A_1}((1/\lambda_1)\, x_1),\ldots, \mu_{A_k}((1/\lambda_k)\, x_k), \mu_{0A_{k+1}}(0),\ldots, \mu_{0A_n}(0)\}]\\
&\quad \leqslant \sup_{x_1+\cdots+x_k=z} \mu_A(\lambda_1(1/\lambda_1)\, x_1 + \cdots + \lambda_k(1/\lambda_k)\, x_k) = \mu_A(z).
\end{aligned}$$

LEMMA 2.7. *Let A, B be fuzzy subsets of E. Then*

(1) $A + 0B \subset A$;

(2) $A + 0B = A$ *iff* $\sup_{x\in E} \mu_A(x) \leqslant \sup_{x\in E} \mu_B(x)$.

Proof. (1) $\mu_A(x + 0y) = \mu_A(x) \geqslant \min\{\mu_A(x), \mu_B(y)\}$. Hence (1) follows from Proposition 2.6.

(2) Suppose that $\sup \mu_A(x) \leqslant \sup \mu_B(x) = \mu_{0B}(0)$. Then

$$\mu_{A+0B}(z) = \sup_{x+y=z} [\min\{\mu_A(x), \mu_{0B}(y)\}] = \min\{\mu_A(z), \mu_{0B}(0)\} = \mu_A(z).$$

On the other hand, if $\mu_A(z) > \sup \mu_B(x) = \mu_{0B}(0)$ for some z, then

$$\mu_{A+0B}(z) = \min\{\mu_A(z), \mu_{0B}(0)\} < \mu_A(z),$$

and hence $A + 0B \neq A$.

3. FUZZY SUBSPACES

DEFINITION. A fuzzy set F in E is called a fuzzy subspace if

(i) $F + F \subset F$;

(ii) $\lambda F \subset F$, for every scalar λ.

LEMMA 3.1. *Let F be a fuzzy set in E. Then, the following are equivalent.*

(1) *F is a subspace of E;*
(2) *For all scalars k, m, we have $kF + mF \subset F$;*
(3) *For all scalars k, m, and all $x, y \in E$, we have*

$$\mu_F(kx + my) \geqslant \min\{\mu_F(x), \mu_F(y)\}.$$

Proof. Clearly, $1 \Rightarrow 2$. Also (2) and (3) are equivalent by (2.6).

$$(2) \Rightarrow (1) \qquad F + F = 1F + 1F \subset F,$$
$$kF = kF + 0F \subset F.$$

PROPOSITION 3.2. *Let E and F be vector spaces over K and let f be a linear map from E into F. If A is a fuzzy subspace of E, then $f(A)$ is a fuzzy subspace of F. Similarly, $f^{-1}(B)$ is a fuzzy subspace of E whenever B is a fuzzy subspace of F.*

Proof. For k, m scalars, we have

$$kf(A) + mf(A) = f(kA + mA) \subset f(A),$$

which shows that $f(A)$ is a fuzzy subspace of F. Also,

$$\begin{aligned} \mu_{f^{-1}(B)}(kx + my) &= \mu_B(f(kx + my)) = \mu_B(kf(x) + mf(y)) \\ &\geqslant \min\{\mu_B(f(x)), \mu_B(f(y))\} \\ &= \min\{\mu_{f^{-1}(B)}(x), \mu_{f^{-1}(B)}(y)\}. \end{aligned}$$

Hence $f^{-1}(B)$ is a fuzzy subspace by 3.1.

We omit the proof of the following easily established proposition.

PROPOSITION 3.3. *If A, B are fuzzy subspaces of E and k is a scalar, then $A + B$ and kA are fuzzy subspaces.*

PROPOSITION 3.4. *If $\{A_i\}_{i\in I}$ is a family of fuzzy subspaces of E, then $A = \bigcap A_i$ is a fuzzy subspace.*

Proof.

$$\begin{aligned} \mu_A(mx + ky) &= \inf_{i\in I} \mu_A\,(mx + ky) \\ &\geqslant \inf_{i\in I}[\min\{\mu_{A_i}(x), \mu_{A_i}(y)\}] \\ &\geqslant \min\{\inf_{i\in I} \mu_{A_i}(x), \inf_{i\in I} \mu_{A_i}(y)\} \\ &= \min\{\mu_A(x), \mu_A(y)\}. \end{aligned}$$

Hence the result follows from 3.1.

4. Convex, Balanced, Absorbing Fuzzy Sets

Definition. A fuzzy set A in E is said to be

(a) Convex if $kA + (1 - k) A \subset A$ for all $k \in [0, 1]$;

(b) Balanced if $kA \subset A$ for all scalars k with $|k| \leqslant 1$;

(c) Absorbing if $E = \bigcup_{k>0} kA$.

We omit the proofs of the next two easily established propositions.

Proposition 4.1. *Let A be a fuzzy set in E. Then, the following assertions are equivalent.*

(1) *A is convex (balanced).*

(2) $\mu_A(kx + (1 - k) y) \geqslant \min\{\mu_A(x), \mu_A(y)\}$ *for all* $x, y \in E$ *and all* $k \in [0, 1]$ ($\mu_A(kx) \geqslant \mu_A(x)$ *for all k with* $|k| \leqslant 1$).

(3) *For each $d \in [0, 1]$, the ordinary set*

$$A_d = \{x \in E: \mu_A(x) \geqslant d\}$$

is convex (balanced).

Proposition 4.2. *For a fuzzy subset A of E, the following are equivalent.*

(1) *A is absorbing.*

(2) *For each $x \in E$,* $\sup_{k>0} \mu_A(kx) = 1$.

(3) *For each d, with $0 \leqslant d \leqslant 1$, the ordinary set*

$$A_d = \{x \in E: \mu_A(x) \geqslant d\}$$

is absorbing.

Proposition 4.3. *Let E, F be vector spaces over K and let $f: E \to F$ be a linear map. If A is a convex (balanced) fuzzy set in E, then $f(A)$ is a convex (balanced) fuzzy set in F. Similarly, $f^{-1}(B)$ is a convex (balanced) fuzzy set in E whenever B is a convex (balanced) fuzzy set in F.*

Proof. We will prove the result for the convex case. The proof for the balanced case is similar. Let $k \in [0, 1]$ and A a convex fuzzy set in E. Then

$$kf(A) + (1 - k) f(A) = f(kA + (1 - k) A) \subset f(A),$$

which proves that $f(A)$ is convex. Assume next that B is a convex fuzzy set in F and let $k \in [0, 1]$. Set

$$M = kf^{-1}(B) + (1 - k) f^{-1}(B).$$

Then,

$$f(M) = kf(f^{-1}(B)) + (1 - k)f(f^{-1}(B)) \subset kB + (1 - k)B \subset B,$$

and hence $M \subset f^{-1}(B)$.

One can also prove easily the following proposition.

PROPOSITION 4.4. *Let $f: E \to F$ be a linear map and A an absorbing fuzzy set in F. Then $f^{-1}(A)$ is an absorbing fuzzy set in E.*

Using 2.3 we get the following.

PROPOSITION 4.5. *If A, B are convex (balanced) fuzzy sets in E, then $A + B$ is a convex (balanced) fuzzy set in E.*

PROPOSITION 4.6. *If $\{A_i\}_{i\in I}$ is a family of convex (balanced) fuzzy sets in E, then $A = \bigcap A_i$ is a convex (balanced) fuzzy set in E.*

Proof. Let $0 \leqslant d \leqslant 1$. Then

$$\{x \in E: \mu_A(x) \geqslant d\} = \bigcap_{i\in I} \{x \in E: \mu_{A_i}(x) \geqslant d\}.$$

Since the intersection of a family of ordinary convex (balanced) subsets of E is convex (balanced), the result follows from 4.1.

DEFINITION. Let A be a fuzzy set in E. The convex (balanced) hull of A is the intersection of all convex (balanced) sets in E which contain A.

By 4.6, the convex (balanced) hull of A is the smallest convex (balanced) fuzzy set in E which contains A.

PROPOSITION 4.7. *Let A be a fuzzy set in E. Then, the balanced hull of A is the fuzzy set $\bigcup_{|\lambda|\leqslant 1} \lambda A$.*

Proof. It is easy to see that the fuzzy set $B = \bigcup_{|\lambda|\leqslant 1} \lambda A$ is contained in any balanced fuzzy set which contains A. Since $B \supset A$, it suffices to show that B is balanced. Let $a \in K$, $|a| \leqslant 1$, and $x \in E$. Then

$$\mu_B(x) = \sup_{|\lambda|\leqslant 1} \mu_{\lambda A}(x) \leqslant \sup_{|\lambda|\leqslant 1} \mu_{a\lambda A}(ax) \leqslant \sup_{|\lambda|\leqslant 1} \mu_{\lambda A}(ax) = \mu_B(ax).$$

Hence $aB \subset B$, by 2.6.

PROPOSITION 4.8. *Let A be a fuzzy subset of E. Then the convex hull* co(A) *of A is the set*

$$B = \bigcup \left\{\lambda_1 A + \cdots + \lambda_n A: \lambda_i \geqslant 0, \sum \lambda_i = 1\right\}.$$

Proof. It is clear that $A \subset B \subset \mathrm{co}(A)$. We will finish the proof by showing that B is convex. To this end we first observe that $\lambda(\bigcup A_i) = \sum \lambda A_i$ and that

$$\left(\bigcup_{i \in I} A_i\right) + \left(\bigcup_{j \in J} B_j\right) \subset \bigcup_{(i,j) \in I \times J} (A_i + B_j).$$

Let now $a \in [0, 1]$, $\sum_1^n \lambda_i = 1$, $\sum_{j=1}^m k_j = 1$, $\lambda_i \geqslant 0$, $k_j \geqslant 0$. Then

$$a\left(\sum_1^n \lambda_i A\right) + (1 - a)\left(\sum_{j=1}^m k_j A\right)$$
$$= a\lambda_1 A + \cdots + a\lambda_n A + (1 - a)\, k_1 A + \cdots + (1 - a)\, k_m A \subset B,$$

since $\sum_1^n a\lambda_i + \sum_{j=1}^m (1 - a)\, k_j = 1$. It follows that $aB + (1 - a)\, B \subset B$, which completes the proof.

Definition. A fuzzy set A is absolutely convex if it is both convex and balanced. The convex balanced hull of a fuzzy set B in E is the intersection of all absolutely convex fuzzy sets in E which contain B.

One can easily prove the following proposition.

Proposition 4.9. *Let A be a fuzzy set in E. Then the following are equivalent*

(1) *A is absolutely convex.*

(2) *$aA + bA \subset A$ for all scalars a, b, with $|a| + |b| \leqslant 1$.*

(3) *$\mu_A(ax + by) \geqslant \min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in E$ and all scalars a, b with $|a| + |b| \leqslant 1$.*

(4) *For each $d \in [0, 1]$, the ordinary set*

$$A_d = \{x \in E\colon \mu_A(x) \geqslant d\}$$

is absolutely convex.

Using an argument similar to that used in the proof of 4.8, we prove the following proposition.

Proposition 4.10. *The convex balanced hull of a fuzzy set A is the fuzzy set*

$$\bigcup \left(\lambda_1 A + \cdots + \lambda_n A\colon \sum |\lambda_i| \leqslant 1\right).$$

5. Quotient Spaces

Lemma 5.1. *Let F be a fuzzy subspace of E. Then*

(1) *$\mu_F(x) \leqslant \mu_F(0)$ for all $x \in E$;*

(2) *The ordinary set*

$$F_0 = \{x \in E: \mu_F(x) = \mu_F(0)\}$$

is an ordinary subspace of E.

Proof. (1) $0F \subset F$. Thus, by 2.6,

$$\mu_F(0) = \mu_F(0x) \geqslant \mu_F(x)$$

for all $x \in E$.

(2) Let a, b be scalars and let $x, y \in F_0$. Then

$$\mu_F(ax + by) \geqslant \min\{\mu_F(x), \mu_F(y)\} = \mu_F(0).$$

Hence $ax + by \in F_0$ by (1).

DEFINITION. For F a fuzzy subspace of E, we define the quotient space E/F to be the ordinary quotient space E/F_0.

For $x \in E$, we denote by $\hat{x}$ the element of E/F to which x belongs. Let $q = q_F$ denote the quotient map.

PROPOSITION 5.2. *If A is a fuzzy set in E, then*

$$q^{-1}(q(A)) = F_0 + A.$$

Proof. Let $B = q^{-1}(q(A))$. Then, for each $z \in E$,

$$\begin{aligned}\mu_B(z) &= \mu_{q(A)}(\hat{z}) = \sup_{x \in \hat{z}} \mu_A(x) = \sup_{y \in F_0} \mu_A(z - y)\\ &= \sup_{y \in F_0} \mu_{y+A}(z) = \mu_{F_0+A}(z).\end{aligned}$$

6. FUZZY TOPOLOGICAL VECTOR SPACES

DEFINITION. A fuzzy topological vector space is a vector space E equipped with a fuzzy topology such that the two maps

(a) $\phi: E \times E \to E$
$\quad (x, y) \to x + y$

(b) $\psi: K \times E \to E$
$\quad (\lambda, x) \to \lambda x$

are continuous when K has the usual topology and $E \times E$, $K \times E$ are given the product fuzzy topologies.

DEFINITION. A fuzzy topology τ on the vector space E is called translation invariant if $x + A \in \tau$ for all $x \in E$ and all $A \in \tau$.

LEMMA 6.1. *If E is equipped with a translation invariant topology, then each map $f_x\colon E \to E$, $y \to x + y$, is a homeomorphism.*

Let E be a fuzzy topological vector space and let F be a fuzzy subspace. Let $q\colon E \to E/F$ denote the quotient map and consider, on E/F, the quotient fuzzy topology.

PROPOSITION 6.2. *If the topology of E is translation invariant, then q is an open map.*

Proof. Let V be open in E and $A = q(V)$. To prove that A is open in E/F it suffices to show that $q^{-1}(A)$ is open in E. But, by 5.2,

$$q^{-1}(A) = F_0 + V = \bigcup_{x \in F_0} (x + V).$$

Hence $q^{-1}(A)$ is open as a union of fuzzy open sets.

LEMMA 6.3. *Let X_1, Y_1, X_2, Y_2 be fuzzy topological spaces and*

$$f_1\colon X_1 \to Y_1, \qquad f_2\colon X_2 \to Y_2,$$

open maps. Then the map

$$f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2, \qquad (x, y) \to (f_1(x), f_2(y))$$

is open.

Proof. The sets of the form $A \times B$, with A open in X_1 and B open in X_2, form a base for the open sets in $X_1 \times X_2$. Since

$$(f_1 \times f_2)(A \times B) = f_1(A) \times f_2(B),$$

the result follows.

PROPOSITION 6.4. *Let E be a fuzzy topological space whose topology is translation invariant. If F is a fuzzy subspace, then E/F with the quotient fuzzy topology is a fuzzy topological vector space.*

Proof. Let

$$\phi\colon E \times E \to E, \qquad (x, y) \to x + y,$$
$$\phi_F\colon E/F \times E/F \to E/F, \qquad (\hat{x}, \hat{y}) \to \hat{x} + \hat{y}.$$

The following diagram is commutative.

$$\begin{tikzcd} E \times E \arrow[r, "\phi"] \arrow[d, "q \times q"] & E \arrow[d, "q"] \\ E/F \times E/F \arrow[r, "\phi_F"'] & E/F \end{tikzcd}$$

Let W be a fuzzy open set in E/F. Then the set $V = (q \circ \phi)^{-1}(W)$ is an open set in $E \times E$. Since $q \times q$ is open by 6.2 and 6.3, the fuzzy set $V_1 = (q \times q)(V)$ is open in $E/F \times E/F$. But $V_1 = \phi_F^{-1}(W)$. This shows that ϕ_F is continuous. The proof of the continuity of the map

$$\psi_F: K \times E/F \to E/F, \qquad (\lambda, \hat{x}) \to \lambda\hat{x},$$

is similar.

PROPOSITION 6.5. *Let $\{E_i\}_{i \in I}$ be a family of fuzzy topological vector spaces over K and let $E = \prod E_i$ with the product fuzzy topology. Then E is a fuzzy topological vector space.*

Proof. Let $p_i: E \to E_i$ denote the ith projection map and

$$\phi: E \times E \to E, \qquad (x, y) \to x + y$$

$$\phi_i: E_i \times E_i \to E_i, \qquad (x, y) \to x + y.$$

The following diagram is commutative.

$$\begin{tikzcd} E \times E \arrow[r, "\phi"] \arrow[d, "p_i \times p_i"] & E \arrow[d, "p_i"] \\ E_i \times E_i \arrow[r, "\phi_i"'] & E_i \end{tikzcd}$$

Since $p_i \times p_i$ is continuous, the composite $\phi_i \circ (p_i \times p_i)$ is continuous. Since each $p_i \circ \phi$ is continuous, it follows that ϕ is continuous by [10, Theorem 3.1]. Similarly we prove that the function

$$\psi: K \times E \to E, \qquad (\lambda, x) \to \lambda x,$$

is continuous. This shows that E is a fuzzy topological vector space.

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